ARTICLES

Application of the non-Markovian Fokker-Planck equation to the resonant activation of a Josephson junction

Ted Hesselroth*
Department of Physics, Loomis Laboratory, University of Illinois, Urbana, Illinois 61021
(Received 22 January 1992; revised manuscript received 22 March 1993)

The non-Markovian Fokker-Planck equation in the energy variable for a system with a driving force is employed to find the escape rate of a particle in a potential well. The energy-dependent diffusion coefficient is the integral of the spectral densities of the applied force and the velocity of the system. The escape-rate formula is applied to resonant activation of a Josephson junction by a microwave driving current with a good comparision to experimental results.

PACS number(s): 05.40.+j, 74.50.+r

I. INTRODUCTION

A thermal activation rate may be enhanced by a resonant driving force. In particular, the rate of escape from the superconducting to the nonsuperconducting state of a Josephson junction may be enhanced by the application of a driving current induced by a microwave source. Devoret *et al.* [1] have obtained experimental resonance curves of the increase in activation rate versus applied frequency.

Many authors have made calculations of the resonance curve for this experiment, which falls into the low-friction, low-noise category of statistical-mechanics dynamics problems. Larkin and Ovchinnikov [2] presented a quantum-mechanical calculation that was extended by Chow and Ambegaokar [3]. Carmeli and Nitzan applied their Fokker-Planck equation to a harmonic and to a Morse potential [4]. Ivlev and Melnikov have written several papers on the subject [5,6]. Fonesca and Grigolini used the correlation function of the velocity and the continued-fraction approach in their work [7]. Linkwitz and Grabert applied the results of their Ref. [8] to the problem [9].

In this paper I will calculate the resonance curve using a Fokker-Planck equation in the energy variable, with a method that utilizes the spectral densities of the driving force and the dynamical system. The main advantage of this approach is that the spectral densities may be easily calculated and that the energy-frequency relationship of the dynamical system may be included in a straightforward way. Comparison to experiment yields very good results.

II. ESCAPE RATE

For the underdamped case, the Fokker-Planck equation for the energy variable is [10–12,8]

$$\frac{\partial\sigma(E,t)}{\partial t}=\frac{\partial}{\partial E}\langle v^2\rangle_E\left[\gamma+D(E)\frac{\partial}{\partial E}\right]\sigma(E,t)\ , \qquad (1)$$

where $\langle v^2\rangle_E$ is the expectation value of the squared velocity when the system is at energy E, γ is the damping, and $D(E)$ is the energy-dependent diffusion coefficient, given by

$$D(E)=\frac{1}{4\pi}\int_{-\infty}^{\infty}d\omega\, S_f(\omega)\frac{S_v(\omega)}{\langle v^2\rangle_E}\ , \qquad (2)$$

where $S_v(\omega)$ and $S_f(\omega)$ are the spectral densities of the driving force and the velocity. The driving force f includes both the external force $f_0(t)$, and that due to the heat bath. The underdamping leads to dephasing between its oscillations and the driving force [4] which allows the interaction to be modeled adequately by the integral in (2).

One may separate the forces due to the external drive and the heat bath, and use instead a diffusion coefficient.

$$D(E)=\gamma k_B T+D_f(E)\ , \qquad (3)$$

where T is the temperature of the heat bath, k_B is Boltzmann's constant, and $D_f(E)$ is defined by (2) with S_f replaced by S_{f_0}, the spectral density of the external driving force alone.

The very low damping case applies when the escape time is small compared to γ^{-1}. I assume that $D(E)$ is small compared to ΔV, the height of the potential well; that is, that $\gamma k_B T$ and $f_0(t)$ are small. Then the mean first-passage time is given by

$$\tau=\frac{D(0)/\gamma}{\gamma\langle v^2\rangle_{\Delta V}}\exp\int_0^{\Delta V}\frac{\gamma dE'}{D(E')}\ . \qquad (4)$$

Now $D_f(E)$ is assumed to be small compared to $\gamma k_B T$ and $1/D(E)$ is expanded to first order:

$$\tau \approx \frac{D(0)/\gamma}{\gamma \langle v^2 \rangle_{\Delta V}} \exp \int_0^{\Delta V} \frac{\gamma dE'}{\gamma k_B T} \left[1 - \frac{D_f(E)}{\gamma k_B T} \right] \tag{5}$$

$$\approx \frac{1}{\gamma} \frac{k_B T}{\Delta V} e^{\Delta V / k_B T} \exp \left[- \int_0^{\Delta V} \frac{D_f(E) dE}{\gamma (k_B T)^2} \right] , \tag{6}$$

the assumption allowing $D(0) \approx \gamma k_B T$ and having also assumed that $\langle v^2 \rangle_{\Delta V} = \Delta V$. When $D_f(E) = 0$, (6) reduces to Kramer's result [13].

Then, as the escape-rate data are usually presented,

$$-\ln \frac{\tau}{\tau_0} = \int_{k_B T}^{\Delta V} \frac{D_f(E) dE}{\gamma (k_B T)^2} , \tag{7}$$

where τ_0 is the Kramer's mean first-passage time with no driving force. When $D_f(E)$ is a constant independent of E, this is similar to the result of Fonesca and Grigolini [7]. The next task is to calculate $D_f(E)$.

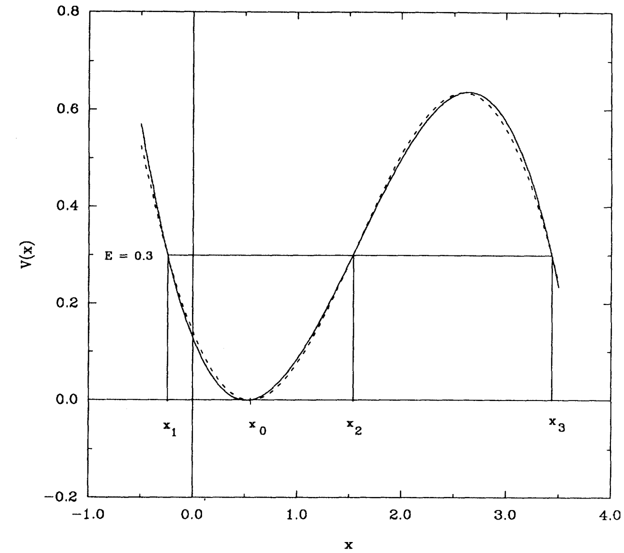

FIG. 1. Plot of potential function vs x. Solid line: $V(x)$ from Eq. (9). Dashed line: approximation from Eq. (11) for $E = 0.3$.

III. CALCULATION OF THE SPECTRAL DENSITIES

A. Dynamical system

The dimensionless equation describing the Josephson junction is

$$\ddot{x} + \gamma \dot{x} + \sin x = \rho_0 + f(t) , \tag{8}$$

where x represents the current through the junction, γ is the damping, ρ_0 is the bias current, and $f(t)$ is the applied driving current. (Now v in the previous section is not the "velocity" but the time derivative of x.)

A cubic approximation to the potential for Eq. (8)

$$V(x) = -(\cos x - \cos x_0) - \rho_0 (x - x_0) , \tag{9}$$

is given by

$$\dot{x}^2 = h^2 (x - x_1)(x - x_2)(x - x_3) , \tag{10}$$

with

$$h^2 = 2E / (x_0 - x_1)(x_0 - x_2)(x_0 - x_3) , \tag{11}$$

where x_1, x_2, and x_3 are given by the values of x for which $V(x) = E$ (Fig. 1), and x_0 is the value of x at the potential minimum. $x(t)$ can be found by standard methods of nonlinear equations [14]. It is

$$x(t) = (x_2 - x_1) \mathrm{sn}^2(hMt, k) + x_1 , \tag{12}$$

where sn is a Jacobi elliptic function and

$$M^2 = \frac{x_3 - x_1}{4} \tag{13}$$

and

$$k^2 = \frac{x_2 - x_1}{x_3 - x_1} , \tag{14}$$

sn(hMt) is an odd function of period $4K/hM$, where $K = K(k)$, the elliptic K function. Then sn(hMt) has the period $2K/hM$. The Fourier expansion of sn(hMt) shows that unless E is extremely close to ΔV, the motion is very sinusoidal. Figure 2 shows the square of the second Fourier coefficient of $\mathrm{sn}^2(hMt)$ in relation to the square of the first as a function of energy for a value of the supply current used in the experiment of Devoret *et al.* [1]. Therefore we can approximate the dynamics of x as occurring in a quadratic potential with angular frequency $\pi h M / K \equiv \omega_E$. Thus [15],

$$\frac{S_v(\omega)}{\langle v^2 \rangle_E} = \frac{2\gamma\omega^2}{(\omega_E^2 - \omega^2)^2 + \gamma^2 \omega^2} . \tag{15}$$

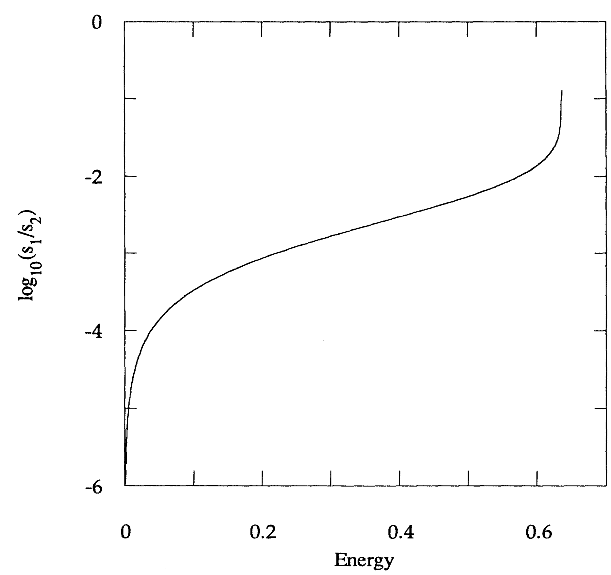

FIG. 2. Plot of the ratio of the square of first Fourier coefficient of $\mathrm{sn}^2(hMt)$, s_1, to the square of the second, s_2, vs energy. Energy is normalized to E_c, defined in Eq. (19). The plot shows that except when its value is very near the barrier height, nearly all of the energy is contained in the first harmonic.

B. Driving force

As for $S_{f_0}(\omega)$, $f_0(t)$ is a sinusoidal driving force of angular frequency Ω. If the amplitude f_0 were constant, then we would have

$$S_{f_0}(\omega)=\frac{f_0^2}{2}\delta(\omega-\Omega) . \tag{16}$$

In the experiment of Devoret *et al.* [1], due to the configuration of the microwave signal, the junction and its loading circuitry are supplied with a constant *voltage.* This is so because the special filter they use between the microwave source and the circuit has very high impedance. Since Eq. (8) is in terms of the current, the frequency dependence of the admittance of the circuit must be taken into account. In terms of the applied voltage V_{ac}

$$S_{f_0}(\omega)=\frac{V_{ac}^2C^2\Omega^2}{I_c^2}\delta(\omega-\Omega) , \tag{17}$$

C is the capacitance of the circuit, which dominates the admittance in an unshunted junction, and I_c is to normalize $f_0(t)$ to the dimensionless units.

IV. CALCULATION OF THE DIFFUSION COEFFICIENT AND THE ESCAPE RATE

Performing the integration over ω for $D_f(E)$, we find

$$D_f(E,\Omega)=\frac{E_c\omega_{pl}^2 2\gamma V_{ac}^2C^2\Omega^4}{4\pi I_c^2[(\omega_E^2-\Omega^2)^2+\gamma^2\Omega^2]} . \tag{18}$$

The factor $E_c\omega_{pl}^2$ has been added to convert the units from dimensionless to real. E_c is the critical energy of the junction, given by

$$\frac{E_c}{I_c}=\frac{\Phi_0}{2\pi} , \tag{19}$$

where $\Phi_0=h/2e=2.07\times10^{-15}$ V s. ω_{pl} is the junction's plasma frequency.

If the capacitance of the loading circuitry is small, we can take C as the capacitance of the junction itself. Then using

$$\omega_{pl}^2=\frac{2\pi I_c}{\Phi_0 C} , \tag{20}$$

Eq. (18) reduces to

$$D_f(E,\Omega)=\frac{\gamma V_{ac}^2C}{2\pi}\frac{\Omega^4}{(\omega_E^2-\Omega^2)^2+\gamma^2\Omega^2} , \tag{21}$$

V_{ac}^2C is the energy stored in the junction.

Now the integral in Eq. (7) can be carried out to find $-\ln(\tau/\tau_0)$. This is done numerically for a value of the source current used in the experiment (Fig. 3). The fit was found by adjusting the frequency and height scales to match the data near the peak, and γ to match the right-hand tail of the data.

The experiment is used to indirectly obtain values for junction parameters. From the fit for $\rho=0.524$, $\omega_{pl}/2\pi$ is

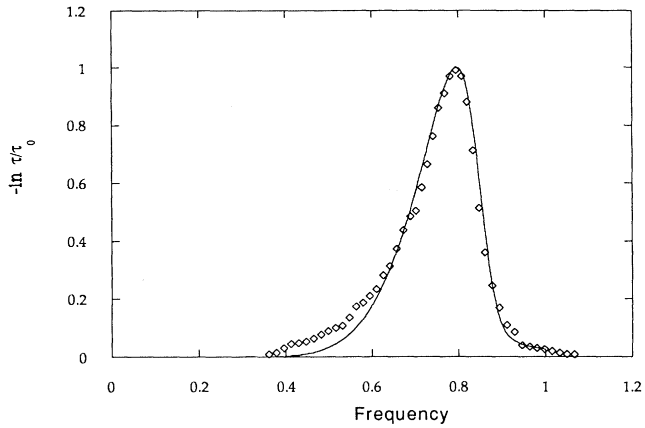

FIG. 3. Plot of normalized mean first-passage time vs frequency for the driven Josephson junction. The frequency is normalized to ω_{pl}. Solid line: Eq. (7) using Eq. (21) with $\gamma=0.02$. Vertical and horizontal scales have also been fitted. Diamonds: experimental results from Fig. 9(a) of Ref. [1]. The normalized dc current is 0.524.

found to be 5.81×10^9 sec^{-1}. This compares with the value found by the experimentalists, 5.87×10^9 sec^{-1}. The value of I_c was stated as 3.09×10^{-6} A. With

$$C=\frac{2\pi I_c}{\Phi_0\omega_{pl}^2} , \tag{22}$$

we get $C=6.96\times10^{-12}$ F.

The fit was not very sensitive to γ, but $\gamma=0.020$ is a reasonable value. Then using

$$\frac{1}{\gamma}=\omega_{pl}RC , \tag{23}$$

we find $R=196\Omega$.

V. DISCUSSION

The main features of this paper are as follows.

(i) Spectral densities were used in the calculation of the energy-dependent diffusion coefficient. The advantage here is that these values are readily known or calculated. Also, on the conceptual level intuition concurs that the amount of diffusion of the system is directly proportional to the amount of overlap between the frequencies of the driving force and the oscillator, as described by Eq. (2). This spectral density method is possible because the low damping of the system leads to dephasing between the driving and system variables.

(ii) The energy-frequency dependence of the oscillator has been taken into account in the spectral densities. The qualitative shape of the resonance curve, with its low-frequency tail and sharp dropoff at higher frequencies, can be understood by the fact that Eq. (8) describes a soft oscillator; its resonant frequency decreases with energy. The low-frequency tail corresponds to higher energies of oscillation, whose corresponding frequencies of oscillation can go all the way to zero, and the cutoff is around the minimum energy of the oscillator, whose corresponding frequency marks the upper limit of the resonant frequencies of the oscillator. Higher applied frequencies do excite the system somewhat, and for a sinusoidal driving force one can show that the zero-energy frequency is at approximately the half maximum of the resonance curve. Below this frequency the resonance curve is described qualitatively by $(d\omega/dE)^{-1}$, where ω is the frequency of the oscillator at energy E. The maximum of the resonance curve is found at the frequency which maximizes this quantity. Intuitively, there are less frequencies here in a given energy range, inhibiting the oscillator from diffusing away to a frequency different from that applied. Thus the oscillator can collect more energy at this applied frequency than others.

(iii) There is an effective frequency dependence of the driving force, due to the admittance of the junction. This affects the shape of the resonance curve at higher frequencies, e.g., the high-frequency tail has become closer to experiment with the inclusion of this effect.

The work presented in this paper was relatively simple because the dynamical system was underdamped, at low temperature, nearly harmonic, and driven by a low-amplitude driving force which was sinusoidal. It is perhaps the simplest application of the method described. The assumptions may be relaxed somewhat, but it is necessary that the damping and the escape rate be small for the spectral density method to be valid.

ACKNOWLEDGMENTS

My thanks to the Center for Complex Systems Research and the Theoretical Biophysics group of the Beckman Institute for Advanced Science for supporting this research.

*Electronic address: ted@lisboa.ks.uiuc.edu

[1] M. H. Devoret, D. Esteve, J. Martinis, A. Cleland, and J. Clarke, Phys. Rev. B **36**, 58 (1987).
[2] A. Larkin and Y. Ovchinnikov, J. Low-Temp. Phys. **63**, 317 (1986).
[3] K. Chow and V. Ambegaokar, Phys. Rev. B **38**, 11 168 (1988).
[4] B. Carmeli and A. Nitzan, Phys. Rev. A **32**, 2439 (1985).
[5] B. Ivlev and V. Melnikov, Zh. Eksp. Teor. Fiz. **89**, 2248 (1985) [Sov. Phys. JETP **62**, 1298 (1985)].
[6] B. Ivlev and V. Melnikov, Zh. Eksp. Teor. Fiz. **90**, 2208 (1986) [Sov. Phys. JETP **63**, 1295 (1986)].
[7] T. Fonesca and P. Grigolini, Phys. Rev. A **33**, 1122 (1986).
[8] S. Linkwitz and H. Grabert, Phys. Rev. B **44**, 11 888 (1991).
[9] S. Linkwitz and H. Grabert, Phys. Rev. B **44**, 11 901 (1991).
[10] B. Carmeli and A. Nitzan, J. Chem. Phys. **79**, 393 (1983).
[11] R. Zwanzig, Phys. Fluids **2**,12 (1959).
[12] R. Grote and J. Hynes, J. Chem. Phys. **77**, 3736 (1982).
[13] H. A. Kramers, Physica **7**, 284 (1940).
[14] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York, 1962).
[15] B. Budgor, K. Lindenberg, and K. E. Shuler, J. Stat. Phys. **15**, 375 (1976).